GENERALIZED HALF-BAND MAXIMALLY FLAT FIR FILTERS

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ABSTRACT

The existence of nonsymmetric generalized half-band lowpass and highpass FIR filters with maximally flat magnitude and group delay characteristics is proved. Like their linear-phase counterparts, approximately half of the impulse response coefficients of these generalized half-band filters are exactly zero. The magnitude response of the filters is not a monotone function in general. However, the filters can be designed to yield improved magnitude response characteristics compared to the linear-phase maximally flat filters. A closed-form formula for the transfer function of the filters is also derived using Bernstein polynomials. The filters may find applications in multirate and wavelet signal processing.

1. INTRODUCTION

A system is called a generalized half-band filter if any one of its two polyphase components is of the form cz^{-n} [2]. As a result, approximately half of the impulse response coefficients of the generalized half-band filters are exactly zero regardless of the linearity of their phase response. In this paper we identify the class of generalized half-band maximally flat (GHBMF) FIR filters. These are nonsymmetric counterparts of the well-known linear-phase maximally flat half-band filters [3]. The frequency response of GHBMF filters approximates an ideal complex-valued function in a maximally flat sense at the frequencies $\omega = 0$ and $\omega = \pi$. The ideal frequency response is chosen to be a linear-phase function with integer-valued phase slope in the passband. This will ensure that the group delay of the resulting filter approximates an integer number of samples in the passband. An approximation problem is formed by incorporating these requirements and imposing a further constraint that the degrees of flatness at the two endpoint frequencies take specific values. We will show that under these constraints, the resulting filter is a generalized half-band system having a maximally flat magnitude response and group delay. Specifically, we will prove that all impulse response coefficients with odd indices are equal to zero except one coefficient whose value is equal to $\frac{1}{2}$. A closed-from expression for the transfer function of GHBMF filters is also derived. This is done using a Theorem that relates the generalized half-band property, the number of zeros of the transfer function at $z = -1$ and the flatness of the magnitude response. The Bernstein polynomials [7, 8] are used to express the transfer function in the closed form.

The proposed approximation method does not distinguish between the magnitude response and group delay. However, it turns out that both the magnitude response and the group delay approximate the ideal response in a maximally flat sense. Nonsymmetric FIR filters with maximally flat magnitude response and group delay have been studied by Baher [4] and, Selesnick and Burrus [5]. Baher derived a closed-form solution for the case where the magnitude response and the group delay have degrees of flatness that differ at most by one. Selesnick and Burrus generalized those filters by subjecting the magnitude and group delay responses to differing number of flatness constraints. The above approaches are concerned with the general case where the group delay may take arbitrary real values (integer or non-integer) at $\omega = 0$, or the degrees of flatness at the two endpoint frequencies may be set to arbitrary numbers. The existence of half-band solutions is not considered in those approaches. This paper not only proves the existence of such solutions, but also gives a very simple formula for their transfer functions.

2. FORMULATION AND STATEMENT OF THE PROBLEM

Let $H(z) = \sum_{i=0}^{N} h_i z^{-i}$ denote the transfer function of the filter, where $N(= 2M)$ is an even integer. $H(z)$ is a system with nonsymmetric impulse response in general and can be decomposed into the sum of a symmetric type 1 and an anti-symmetric type 2 linear-phase subsystem, i.e.,

$$H(z) = H_s(z) + H_a(z), \tag{1}$$

where

$$\begin{aligned} H_s(z) &= \tfrac{1}{2}\left(H(z) + z^{-N}H(z^{-1})\right), \\ H_a(z) &= \tfrac{1}{2}\left(H(z) - z^{-N}H(z^{-1})\right). \end{aligned}$$

The frequency response of the filter is given by

$$H(e^{j\omega}) = e^{-jM\omega}\,\left(H_{s0}(\omega) + jH_{a0}(\omega)\right), \tag{2}$$

where [2]

$$\begin{aligned} H_{s0}(\omega) &= \sum_{n=0}^{M} a_n \cos(n\omega), \\ H_{a0}(\omega) &= \sin\omega \sum_{n=0}^{M-1} b_n \cos(n\omega). \end{aligned}$$

We define the ideal frequency response

$$I(\omega) = \begin{cases} e^{-j(M+d)\omega} & \dots \quad \omega \in \text{Passband} \\ 0 & \dots \quad \omega \in \text{Stopband} \end{cases} \tag{3}$$

where d is a positive or negative integer. In this paper, a maximally flat approximation to $I(\omega)$ is denoted as

$$\begin{array}{rcl} H(e^{j\omega}) & \overset{0,L_0}{\approx} & e^{-j(M+d)\omega}, \\ H(e^{j\omega}) & \overset{\pi,L_\pi}{\approx} & 0. \end{array} \tag{4}$$

The notation $L.H.S. \overset{\omega_0,L_{\omega_0}}{\approx} R.H.S.$ denotes that $L.H.S. - R.H.S.$ has $L_{\omega_0} - 1$ vanishing derivatives at $\omega = \omega_0$, i.e.,

$$L.H.S. - R.H.S. = \mathcal{O}\left((\omega - \omega_0)^{L_{\omega_0}}\right).$$

The term $\mathcal{O}\left((\omega - \omega_0)^{L_{\omega_0}}\right)$ represents omitted higher-order terms in the power series expansion of $L.H.S. - R.H.S.$ about ω_0.

Equation (4) is equivalent to

$$\begin{cases} H_{s0}(\omega) - \cos(d\omega) & \overset{0,L_{0,s}}{\approx} \quad 0, \\ H_{s0}(\omega) & \overset{\pi,L_{\pi,s}}{\approx} \quad 0, \end{cases} \\ \begin{cases} H_{a0}(\omega) + \sin(d\omega) & \overset{0,L_{0,a}}{\approx} \quad 0, \\ H_{a0}(\omega) & \overset{\pi,L_{\pi,a}}{\approx} \quad 0. \end{cases} \tag{5}$$

To simplify mathematical derivations, we use the well-known transformation

$$x = \cos\omega$$

and set

$$\begin{array}{l} L_{0,s} = L_{\pi,s} = 2P \\ L_{0,a} = L_{\pi,a} = 2Q \end{array} \tag{6}$$

to arrive at the more specific formulation

$$\begin{cases} H_{s0}(x) & \overset{+1,P}{\approx} \quad T_d(x), \\ H_{s0}(x) & \overset{-1,P}{\approx} \quad 0, \end{cases} \\ \begin{cases} H_{a0}(x) & \overset{+1,Q}{\approx} \quad -\sqrt{1-x^2}U_{d-1}(x), \\ H_{a0}(x) & \overset{-1,Q}{\approx} \quad 0. \end{cases} \tag{7}$$

in the x domain. Here $T_d(x)$ represents the dth order Chebyshev polynomial of the first kind and $U_{d-1}(x)$ is the $d-1$th order Chebyshev polynomial of the second kind [6]. Considering the power series expansion of $\cos(\omega)$ at $\omega = 0$ and $\omega = \pi$, it can be easily understood that a solution to (5) in the frequency domain is equivalent to a solution to (7) in the x domain. To obtain a maximally flat solution, all available degrees of freedom should be used to maximize the degrees of flatness P and Q. Notice that the degrees of flatness at $x = 1$ and $x = -1$ are identical in each subproblem. The relationship between N and the degrees of flatness will be clarified later.

3. SOLUTION AND ITS POLYPHASE DECOMPOSITION

In this section, we first derive a general analytic solution to the approximation problem (7). We then prove that this solution corresponds to a generalized half-band filter.

3.1. General Analytic Solution

To determine the actual form of functions $H_{s0}(x)$ and $H_{a0}(x)$, we take an approach based on decomposition of polynomials into even and odd power terms. To this end, we write

$$\begin{array}{l} H_{s0}(x) = H_{se}(x) + H_{so}(x) \\ H_{a0}(x) = \sqrt{1-x^2}\,(H_{ae}(x) + H_{ao}(x)) \end{array}$$

$H_{se}(x)$ and $H_{ae}(x)$ are even polynomials, while $H_{so}(x)$ and $H_{ao}(x)$ are odd polynomials. Substituting the above relation into (7) and solving the resulting linear system of equations, we find that

$$\begin{array}{rcl} H_{se}(x) & \overset{+1,P}{\approx} & \frac{1}{2}T_d(x), \\ H_{so}(x) & \overset{+1,P}{\approx} & \frac{1}{2}T_d(x), \\ H_{ae}(x) & \overset{+1,Q}{\approx} & -\frac{1}{2}U_{d-1}(x), \\ H_{ao}(x) & \overset{+1,Q}{\approx} & -\frac{1}{2}U_{d-1}(x). \end{array} \tag{8}$$

In this paper we consider the solution to (8) under the constraint that $|d| < M$, and $M+d$ being an odd integer. In this case, we can always obtain exact solutions (with infinite degrees of flatness) for two of the four approximations in (8). Specifically, for odd M the solution is of the form

$$\begin{array}{l} H_{se}(x) = \frac{1}{2}T_d(x), \\ H_{so}(x) = \sum_{i=0}^{\frac{M-1}{2}} c_{2i+1}^{(so)} x^{2i+1}, \\ H_{ae}(x) = \sum_{i=0}^{\frac{M-1}{2}} c_{2i}^{(ae)} x^{2i}, \\ H_{ao}(x) = -\frac{1}{2}U_{d-1}(x), \end{array} \tag{9}$$

and for even M we find that

$$\begin{array}{l} H_{se}(x) = \sum_{i=0}^{\frac{M}{2}} c_{2i}^{(se)} x^{2i}, \\ H_{so}(x) = \frac{1}{2}T_d(x), \\ H_{ae}(x) = -\frac{1}{2}U_{d-1}(x), \\ H_{ao}(x) = \sum_{i=0}^{\frac{M}{2}-1} c_{2i+1}^{(ao)} x^{2i+1}. \end{array} \tag{10}$$

The coefficients $c_{2i+1}^{(so)}$, $c_{2i}^{(ae)}$, $c_{2i}^{(se)}$, $c_{2i+1}^{(ao)}$, should be chosen so that the maximum number of vanishing derivatives be obtained when the polynomials are substituted to the left-hand sides of (8). This can be done by forming and solving systems of linear equations with the corresponding coefficients as unknowns. An explicit solution is given in Section 4.

3.2. Polyphase Decomposition of Solution

Using the relation $x = \frac{z+z^{-1}}{2}$, the overall transfer function for odd M is expressed as

$$H(z) = z^{-M}\left(\frac{1}{2}T_d(\tfrac{z+z^{-1}}{2}) + \sum_{i=0}^{\frac{M-1}{2}} c_{2i+1}^{(s0)}\left(\frac{z+z^{-1}}{2}\right)^{2i+1} + \right. \\ \left. \frac{z-z^{-1}}{2}\left(\sum_{i=0}^{\frac{M-1}{2}} c_{2i}^{(ae)}\left(\frac{z+z^{-1}}{2}\right)^{2i} - \frac{1}{2}U_{d-1}(\frac{z+z^{-1}}{2})\right)\right)$$

Simple algebraic manipulations reveal that

$$H(z) = \frac{1}{2}z^{-M-d} + E(z^2) \tag{11}$$

Table 1: Degrees of flatness at $\omega = 0$ and $\omega = \pi$ for magnitude response and group delay of GHBMF filters.

M	$\|H(\omega)\|_{\omega=0}$	$\tau(\omega)_{\omega=0}$	$\|H(\omega)\|_{\omega=\pi}$	$\tau(\omega)_{\omega=\pi}$
Even	$1+\mathcal{O}(\omega^{M+2})$	$M+d+\mathcal{O}(\omega^{M})$	$0+\mathcal{O}\left((\omega-\pi)^{M+1}\right)$	$\tau(\pi)+\mathcal{O}\left((\omega-\pi)^2\right)$
Odd	$1+\mathcal{O}(\omega^{M+1})$	$M+d+\mathcal{O}(\omega^{M+1})$	$0+\mathcal{O}\left((\omega-\pi)^{M+1}\right)$	$\tau(\pi)+\left(\mathcal{O}(\omega-\pi)^2\right)$

where d is an even integer and $E(z^2)$ is an even polynomial of degree $2M$ in z^{-1}. In other words, $H(z)$ is a generalized half-band filter. For even M, it can be shown in a similar manner that the the transfer function can be decomposed as (11), where d is an odd integer. In summary, we proved that a solution to (7) gives rise to a generalized half-band filter provided that the parity of $M+d$ is odd and $|d| < M$. It is interesting to note that half-band solutions do not exist when $|d| \geq M$ or $M+d$ is an even number.

3.3. Relation to Linear-Phase Filters

The class of GHBMF filters includes the linear-phase maximally flat filters as a special case. For linear-phase maximally flat filters, M is an odd number and $d = 0$. It can be easily verified that under this condition, the polynomials $H_{ae}(x)$ and $H_{a0}(x)$ are identically zero, and thus the solution is a type 1 linear-phase filter.

3.4. Magnitude Response and Group Delay

The degrees of flatness of the magnitude response and group delay of GHBMF filters are given in Table 1. Unlike linear-phase maximally flat filters the nonlinear-phase counterparts do not possess a monotone magnitude response in general. However, monotone characteristics may be obtained for some special values of parameters M and d. Figure 1 depicts the frequency responses of five GHBMF filters of order $N = 38$ ($d = 0, \pm 2, \pm 4$). Note that the magnitude response depends on the absolute value of d, while the group delay is also dependent on the signature of d. The group delay approximation is exact for the linear-phase case corresponding to $d = 0$.

Our study shows that for a fixed filter length, the magnitude response exhibits larger overshoots in the passband as the value of $|d|$ increases. The overshoot is negligible for small values of $|d|$ and may not be tolerable when $|d|$ is relatively large with respect to M. However, improvement in frequency selectivity compared to the linear-phase case can be observed.

4. CLOSED-FORM SOLUTION

The following Theorem can be proved.

Theorem *Any generalized real half-band filter of order $N = 2M$ with $M+1$ multiple zeros at $z = -1$ is a GHBMF filter.*

This Theorem states that for a generalized half-band transfer function, having $M+1$ zeros at $z = -1$ guarantees maximum possible number of vanishing derivatives at $\omega = 0$. In other words, like the linear-phase case, we do not need to impose flatness constraints on both passband and stopband; constructing a flat stopband is sufficient. This property does not generally hold for other optimality criteria regarding generalized half-band filters.

Using the above Theorem and Bernstein polynomials, a closed-form expression for the transfer function of GHBMF filters can be derived. Let

$$H(z) = \sum_{i=0}^{N} b_i \binom{N}{i} \left(\frac{1+z^{-1}}{2}\right)^i \left(\frac{1-z^{-1}}{2}\right)^{N-i} \tag{12}$$

where $\binom{N}{i}$ denotes the binomial coefficients. The right-hand side of (12) is called the Bernstein form of the transfer function. The coefficients b_i should be chosen so that

(a) $H(z)$ has $M+1$ zeros at $z = -1$.
(b) $H(z)$ is a generalized half-band filter, i.e., $H(z) - H(-z) = z^{-M-d}$.

To satisfy condition (a), we simply set

$$b_i = 0, \qquad i = 0, \ldots, M. \tag{13}$$

Condition (b) should be used to determine the remaining Bernstein coefficients. We have

$$H(z) - H(-z) = \sum_{i=0}^{N} (b_i - b_{N-i}) \binom{N}{i} \left(\frac{1+z^{-1}}{2}\right)^i \left(\frac{1-z^{-1}}{2}\right)^{N-i}.$$

On the other hand, it can be proved that

$$z^{-M-d} = \sum_{i=0}^{N} \left(\sum_{j=0}^{i} \frac{(-1)^{M+d-i+j} \binom{M-d}{j} \binom{M+d}{i-j}}{\binom{N}{i}} \right) \binom{N}{i} \left(\frac{1+z^{-1}}{2}\right)^i \left(\frac{1-z^{-1}}{2}\right)^{N-i}.$$

Thus we find that

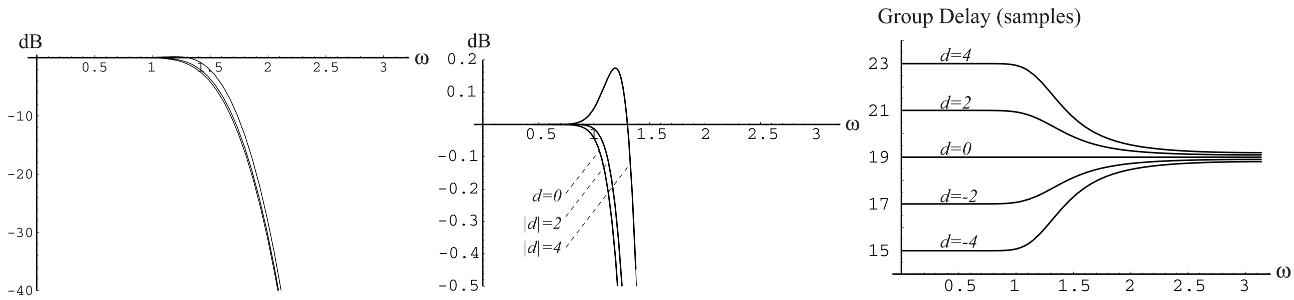
$$b_i = \sum_{j=0}^{i} \frac{(-1)^{M+d-i+j} \binom{M-d}{j} \binom{M+d}{i-j}}{\binom{N}{i}}, \quad i = M+1, M+2, \ldots, 2M (= N). \tag{14}$$

Consequently, the transfer function can be expressed as

$$H_{M,d}(z) = \left(\frac{1+z^{-1}}{2}\right)^{M+1} \sum_{i=0}^{M-1} c_{i+M+1} \left(\frac{1+z^{-1}}{2}\right)^i \left(\frac{1-z^{-1}}{2}\right)^{M-1-i} \tag{15}$$

where

$$c_i = \sum_{j=0}^{i} (-1)^{M+d-i+j} \binom{M-d}{j} \binom{M+d}{i-j}. \tag{16}$$



(a) (b) (c)

Figure 1: Frequency responses for $N = 38$, $d = 0, \pm 2, \pm 4$. (a) Magnitude response. (b) Passband details. (c) Group delay.

Table 2: Impulse response $\{h_i\}$ of GHBMF filters.

N	d	$\{h_i\} \quad i = 0, \ldots, 2M$
2	0	$\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$
4	-1	$\{\frac{3}{16}, \frac{1}{2}, \frac{3}{8}, 0, \frac{-1}{16}\}$
6	0	$\{\frac{-1}{32}, 0, \frac{9}{32}, \frac{1}{2}, \frac{9}{32}, 0, \frac{-1}{32}\}$
6	-2	$\{\frac{5}{32}, \frac{1}{2}, \frac{15}{32}, 0, \frac{-5}{32}, 0, \frac{1}{32}\}$
8	-1	$\{\frac{-5}{256}, 0, \frac{15}{64}, \frac{1}{2}, \frac{45}{128}, 0, \frac{-5}{64}, 0, \frac{-3}{256}\}$
10	0	$\{\frac{3}{512}, 0, \frac{-25}{512}, 0, \frac{75}{256}, \frac{1}{2}, \frac{75}{256}, 0, \frac{-25}{512}, 0, \frac{3}{512}\}$
10	-2	$\{\frac{-7}{512}, 0, \frac{105}{512}, \frac{1}{2}, \frac{105}{256}, 0, \frac{-35}{256}, 0, \frac{21}{512}, 0, \frac{-3}{512}\}$

Table 3: Coefficients $\{c_{i+M+1}\}$ of GHBMF filters.

N	d	$\{c_{i+M+1}\} \quad i = 0, \ldots, M-1$
2	0	$\{1\}$
4	-1	$\{2, 1\}$
6	0	$\{-3, 0, 1\}$
6	-2	$\{5, 4, 1\}$
8	-1	$\{-6, -2, 2, 1\}$
10	0	$\{10, 0, -5, 0, 1\}$
10	-2	$\{-14, -8, 3, 4, 1\}$

Equation (15) is a closed-form solution of the approximation problem stated in Sections 2 and 3. It provides a unified formula for the transfer functions of linear-phase and nonlinear-phase half-band maximally flat FIR filters. Application of Bernstein polynomials to derive closed-form expressions for linear-phase maximally flat filters was first proposed in [8]. Equation (15), can also be used as an alternative expression for the linear-phase half-band case ($d = 0$). Coefficients of some low order GHBMF filers are given in Tables 2 and 3.

5. CONCLUSIONS

A class of nonsymmetric half-band maximally flat filters was identified. It was shown that a certain maximally flat approximation to an ideal frequency response with linear phase and integral group delay in the passband results in nonsymmetric half-band FIR filters. The group delay of GHBMF filters approximates an integer value in the passband and their magnitude response is maximally flat at frequencies $\omega = 0$ and $\omega = \pi$. The filters include the linear-phase maximally flat filters as a special case and are parametrized by M, a parameter that is equal to half of the order of the filter, and d, a parameter that, together with M, specifies the group delay at $\omega = 0$.Compared to the linear-phase counterparts, nonsymmetric GHBMF filters enjoy narrower transition band-widths and may be designed to exhibit lower group delay in the passband. The magnitude response of GHBMF filters is not monotone in general. This is due to overshoots that occur near passband edge frequencies. The magnitude of the overshoots is negligible for the cases where $|d|$ is small relative to M, and may be large enough to render the filter useless for larger values of $|d|$. GHBMF filters provide solutions where no linear-phase half-band filters exist. For example 5- and 9-tap GHBMF filters with $d = \pm 1$ possess monotone magnitude response characteristics and fill the gap among 3-, 7- and 11-tap linear-phase half-band filters. We used Bernstein polynomials to derive a closed-form formula for the transfer function of GHBMF filters.

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