Universal Maximally Flat Lowpass FIR Systems

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***Abstract*—The family of FIR digital filters with maximally flat magnitude and group delay response is considered. The filters were proposed by Baher, who furnished them with an analytic procedure for derivation of their transfer function. The contributions of this paper are the following. A simplified formula is presented for the transfer function of the filters. The equivalence of the novel formula with a formula that is derived from Baher's analytical procedure is proved using a modern method for automatic proof of identities involving binomial coefficients. The universality of Baher's filters is then established by proving that they include linear-phase filters, generalized half-band filters, and fractional delay systems. In this way, several classes of maximally flat filters are unified under a single formula. The generating function of the filters is also derived. This enables us to develop multiplierless cellular array structures for exact realization of a subset of the filters. The subset that enjoys such multiplierless realizations includes linear-phase filters, some nonsymmetric filters, and generalized halfband filters. A procedure for designing the cellular array structures is also presented.**

***Index Terms*—FIR digital filters, fractional delay systems, generating function, interpolation, maximally flat magnitude filters, multiplierless implementation, systolic arrays.**

I. Introduction

WE START by giving a brief account of explicit formulas and analytic techniques for derivation of transfer functions of maximally flat FIR digital filters and then clarify the object of this paper. As is clear from the conventional usage of the term "maximally flat" in the literature, we are not concerned with explicit formulas for filters that have some of their degrees of freedom assigned to purposes other than vanishment of frequency response derivatives. Transfer functions of lowpass and highpass FIR digital filters of even order with exact linear phase and maximally flat magnitude response characteristics are among the most well-known cases of optimal digital filters with closed-form formulas. The transfer functions can be expressed using the celebrated formula of Herrmann [1] or other equivalent formulas [2], [3], [6]. Closed-form formulas are also available for filters of odd order and bandpass filters as well [8], [9]. Nonlinear-phase maximally flat FIR filters were first introduced by Baher [5], who relaxed the constant group delay requirements of Herrmann's filters by imposing simultaneous flatness constraints on the magnitude and group delay only at the frequency $\omega = 0$. This, in turn, resulted in improved transition bandwidth and controllable group delay at $\omega = 0$. Although Baher provides a simple procedure for determining the transfer function, he does not give a compact formula in [5]. Selesnick and Burrus [7] generalized Baher's approach by subjecting the magnitude and group delay responses to differing number of flatness constraints. Their approach is based on the computation of Gröbner bases, while they offer an analytic design technique for a special case. The passband group delay can be specified by the designer in both the Baher and the Selesnick and Burrus approaches. On the other hand, the theory of maximally flat FIR fractional delay systems, which are also known as Lagrange interpolators, has been developed independently. Various approaches toward Lagrange interpolators and other fractional delay systems are listed in [10]. Since the ideal frequency response of fractional delay systems is of allpass form, the connection between Lagrange interpolators and lowpass maximally flat filters is not intuitive. Hence, to our knowledge, there have been no studies in the literature on a possible relation between the two maximally flat systems.

In this paper, we unify Baher's nonsymmetric filters, linear-phase maximally flat filters, and Lagrange interpolators using a novel compact formula for the transfer function of Baher's filters. We develop a simple explicit expression for the class of FIR systems denoted $H_{N,K,d}(z)$ that contains all the aforementioned filters. The expression is parametrized by the number of zeros at $z = -1$, K, the order of the system N, and a parameter d that is related to the value of group delay at $\omega = 0$. We also specify the values of parameters N, K, and d that result in halfband solutions. The halfband solutions, bearing important implications for design of regular wavelets, include the well-known linear-phase halfband filters [4] and the recently developed nonsymmetric halfband filter [11]. Furthermore, a generating function is derived for the entire family of filters. The generating function enables us to develop multiplierless cellular array structures for filters having integer values of $N/2 + d$. We present a versatile cellular array structure for exact multiplierless realization of all linear- and nonlinear-phase maximally flat FIR filters with integer values of $N/2 + d$.

The rest of the paper is organized as follows. In Section II, the result of analytic design procedure of Baher is expressed as a compact formula, and its equivalence with a simpler general formula is established. In Section III, it is shown that the Lagrange interpolator and linear-phase lowpass solutions are, in fact, special cases of the general formula. A generating function for the filters is developed in Section IV, and multiplierless cellular array realizations for some special cases are also given. Conclusions are drawn in Section V.

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II. Simplification of Baher's Formula

Following the procedure for derivation of the transfer function of Baher's Nth-order maximally flat FIR filters, as described in [5], the explicit form of the transfer function can be written as

$$H(z) = \sum_{j=0}^{N-K} b_j \left(\frac{1-z^{-1}}{2}\right)^j \left(\frac{1+z^{-1}}{2}\right)^{N-j}. \tag{1}$$

This expression is related to the Bernstein form of the polynomial $H(z)$ [12], [13]. The coefficients b_j are given by

$$b_j = \sum_{k=0}^{j} \binom{N}{k} A_{j-k} \tag{2}$$

where

$$A_r = (-1)^r \frac{\alpha}{r!} \sum_{i=0}^{r} \binom{r}{i} \frac{\Gamma(\alpha+r-i)}{\Gamma(\alpha-i+1)}. \tag{3}$$

The parameter α controls the group delay $\tau(\omega)$ of the filter via the relation $\alpha = \tau(0)$. Combining the above formulas, the transfer function becomes

$$\begin{aligned} H_{N,K,\alpha}(z) = \left(\frac{1+z^{-1}}{2}\right)^K & \sum_{j=0}^{N-K} \sum_{k=0}^{j} \binom{N}{k} \\ & \cdot \frac{(-1)^{j-k}\alpha}{(j-k)!} \sum_{i=0}^{j-k} \binom{j-k}{i} \\ & \cdot \frac{\Gamma(\alpha+j-k-i)}{\Gamma(\alpha-i+1)} \\ & \cdot \left(\frac{1-z^{-1}}{2}\right)^j \left(\frac{1+z^{-1}}{2}\right)^{N-K-j}. \end{aligned} \tag{4}$$

Once the order of the filter N, the number of zeros at $z=-1$, K, and the value of group delay at $\omega = 0$, α are specified, the transfer function is uniquely determined by (4). Notations like $H_{N,K,\alpha}(z)$ will be used throughout this paper to emphasize the dependence of the filter on the three parameters and provide the actual parameter values whenever necessary. The closed-form formula of (4) can be used to compute the impulse response coefficients h_i of the filter or the values of b_j in the Bernstein-form representation (1). However, any attempt to unfold the relationship between Baher's filters and other maximally flat filters is hindered by presence of three-fold nested summations in combination with Gamma functions and binomial coefficients. However, the expression for b_j can be represented in a simpler form. First, let us define a new delay parameter

$$d = \alpha - \frac{N}{2} \tag{5}$$

and a sequence of numbers c_j as

$$c_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{\frac{N}{2}-d}{i} \binom{\frac{N}{2}+d}{j-i}. \tag{6}$$

For noninteger values of d, the binomial coefficients involved in the above expression are evaluated using

$$\binom{x}{i} = \begin{cases} \prod_{j=0}^{i-1} \frac{x-j}{j+1}, & i \geq 1 \\ 1, & i = 0 \\ 0, & i < 0. \end{cases} \tag{7}$$

Now we assert the following.

Theorem 1: $c_j = b_j$ for all integers j.

To establish the equivalence of (2) and our simplified formula (6), we note that both formulas are sums of hypergeometric terms. Recently, a systematic method has been developed for automatic proof of identities involving binomial coefficients on computers. The method has its origin in the work of Sister Mary Celine Fasenmyer, who showed "how recurrences for certain polynomial sequences could be found algorithmically" (see [15] and references therein).

Proof: To prove that $b_j = c_j$, we used a computer algebra package that generates computer proofs of hypergeometric multisum identities [16]. The package successfully provided a recurrence relation of the form

$$jS_j + 2dS_{j-1} - (j-N-2)S_{j-2} = 0 \tag{8}$$

that is satisfied by the formulas for both b_j and c_j. Furthermore, it can be easily verified (by hand calculation or a computer algebra system) that $b_0 = c_0 = 1$ and $b_j = c_j = 0$ for $j \leq -1$. This means that under the initial conditions of $S_0 = 1$ and $S_j = 0$, $j \leq -1$, running the above recurrence formula for $j \geq 1$, we get a sequence that is identical to the values generated by b_j and c_j. This completely establishes that $b_j = c_j$ for all integers j. ∎

From (6), it can be observed that coefficients c_j take on rational values for rational values of d. Furthermore

$$c_j(-d) = (-1)^j c_j(d) \tag{9}$$

that is the magnitude of c_j is independent of the sign of d. The plots given in Fig. 1 help us to get a rough grasp of the effect of parameter d, or α equivalently, on the magnitude response and group delay. Unlike linear-phase lowpass filters, a monotonic magnitude response is no longer guaranteed. The plots are provided for negative values of d. Such choices of d result in reduced group delay in the passband. The magnitude response is invariant to the sign of d, and the plots of Fig. 1(a) and (b) are valid for both positive and negative values of d. Invoking (9), we find that changing the sign of d amounts to reversing the sign of parameters c_j with odd indexes. In a Bernstein-form representation, this results in a transfer function of the form $z^{-N}H(z^{-1})$, whose magnitude response is identical to that of $H(z)$. In short, we can generally write

$$H_{N,K,-d}(z) = z^{-N} H_{N,K,d}(z^{-1}).$$

Consequently, the group delay $\tau_{N,K,d}(\omega)$ satisfies the relationship

$$\tau_{N,K,-d}(\omega) = N - \tau_{N,K,d}(\omega).$$

The above symmetric relations involving the transfer function and group delay are desirable conveniences afforded by the newly defined parameter d.

III. Universality of Baher's Filters

Having derived the simplified expression

$$\begin{aligned} H_{N,K,d}(z) = \left(\frac{1+z^{-1}}{2}\right)^K & \sum_{j=0}^{N-K} \sum_{i=0}^{j} \\ & \cdot (-1)^{j-i} \binom{\frac{N}{2}-d}{i} \binom{\frac{N}{2}+d}{j-i} \\ & \cdot \left(\frac{1-z^{-1}}{2}\right)^j \left(\frac{1+z^{-1}}{2}\right)^{N-K-j} \end{aligned} \tag{10}$$

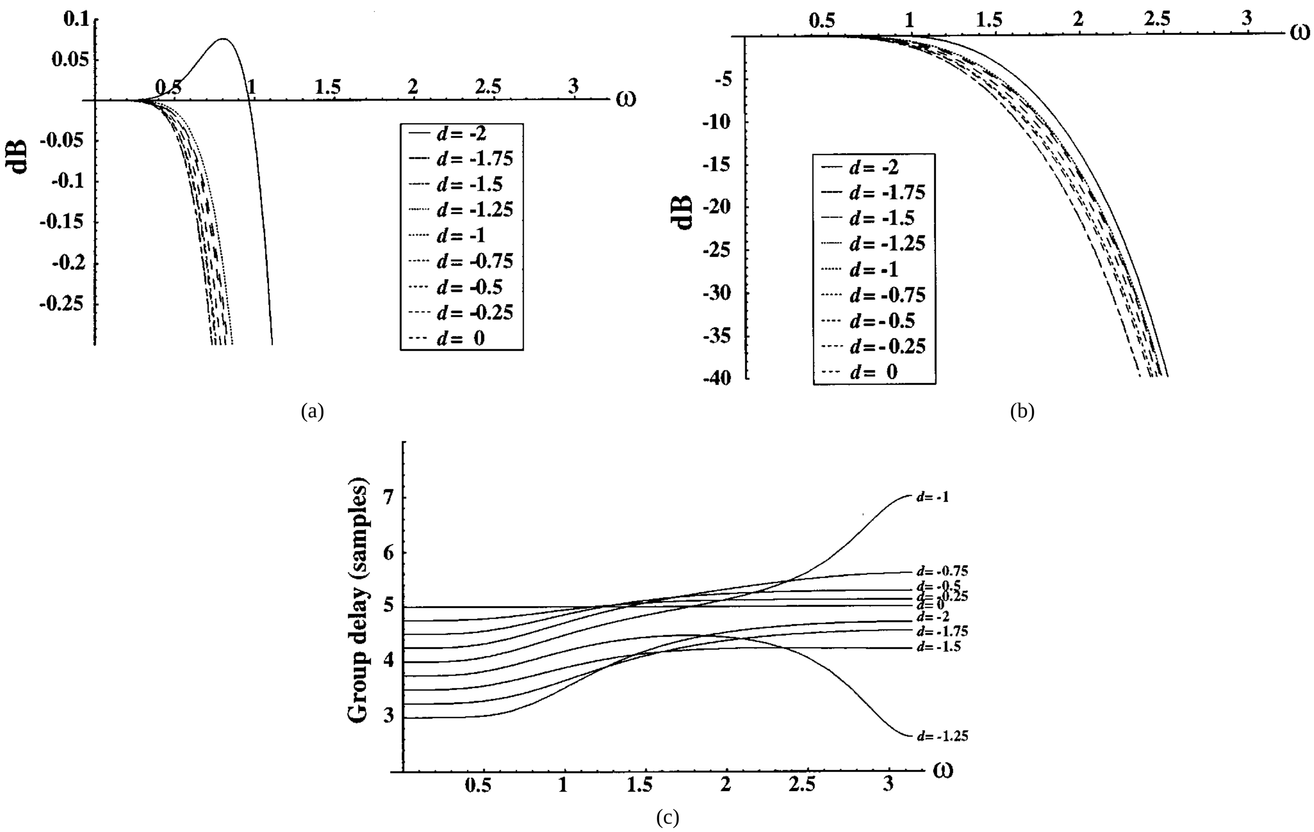


Fig. 1. (a), (b) Magnitude response and (c) group delay for $N = 10$, $K = 6$, $d \in \{-2, -1.75, -1.5, -1.25, -1, -0.75, -0.5, -0.25, 0\}$.

TABLE I
SPECIAL VALUES OF PARAMETERS IN BAHER'S FILTER AND THE FEATURES OF RESULTING SYSTEMS

N	K	d	Other conditions	Resulting system
even		0		Type I linear-phase filter
odd		0		Type II linear-phase filter
$N = 2 \pmod 4$	$\frac{N}{2}+1$	0		Half-band linear-phase filter
even	$\frac{N}{2}+1$	integer	$\frac{N}{2}+d$ odd, $\lvert d\rvert < \frac{N}{2}$	Generalized half-band filter
	0	real		Fractional delay (Lagrange interpolator)

for the transfer function of Baher's filters, we are in a position to establish that for certain choices of the parameters, the formula yields maximally flat linear-phase filters, generalized half-band filters, or Lagrange interpolators. The use of parameter d, which was introduced in the preceding section, instead of the original parameter α used by Baher, is a matter of preference to emphasize the symmetry properties discussed in Section II. Table I presents some special values of the parameters and the corresponding type of the transfer function. It can be seen that Baher's filters include a wide class of maximally flat FIR filters. The proofs and remarks are provided next.

A. N Even, $d = 0$

Setting $d = 0$, the group delay at $\omega = 0$ becomes $N/2$ samples. This is the same situation as a type I linear-phase filter of the same order. Here, we show that, in fact, $H_{N,K,0}(z)$ is a linear-phase filter. Substituting $d = 0$ into (6), we have

$$c_j|_{d=0} = \sum_{i=0}^{j} (-1)^{j-i} \binom{\frac{N}{2}}{i} \binom{\frac{N}{2}}{j-i}. \tag{11}$$

As is shown in [14], using the binomial expansion theorem, it can be easily seen that c_j is the coefficient of the jth power of x in the power series expansion of the product

$$(1-x)^{N/2}(1+x)^{N/2}.$$

Alternatively, the above product can be expanded as

$$(1-x^2)^{N/2} = \sum_{j=0}^{N/2} (-1)^j \binom{\frac{N}{2}}{j} x^{2j}. \tag{12}$$

This yields the relationship

$$c_j = \begin{cases} (-1)^{j/2} \binom{\frac{N}{2}}{\frac{j}{2}}, & j \text{ even} \\ 0, & j \text{ odd.} \end{cases} \tag{13}$$

Hence, the transfer function can be expressed using the binomial coefficients $\binom{\frac{N}{2}}{j}$ as in (14), shown at the bottom of the next page. This means that for $d = 0$, only filters with an even number of multiple zeros at $z = -1$ are obtained. The right-hand side of (14) is identical to Miller's formula for linear-phase maximally flat filters [2], [6].

B. N odd, $d = 0$

Here, we show that for odd values of N, the choice of $d = 0$ results in linear phase. This means that Baher's filters cover linear-phase maximally flat filters of odd order as well. We show that $H_{N,K,0}(z)$ is a symmetric polynomial in z^{-1} for all integer values of N. First, note that a Bernstein-form polynomial in z^{-1}

$$F(z) = \sum_{j=0}^{N} a_j \binom{N}{j} \left(\frac{1-z^{-1}}{2}\right)^j \left(\frac{1+z^{-1}}{2}\right)^{N-j} \tag{15}$$

is symmetric if and only if $a_j = 0$, j odd. This can be readily verified by subjecting $F(z)$ to the condition

$$F(z) - z^{-N} F(z^{-1}) = 0 \tag{16}$$

and writing down the resulting set of linear equations for coefficients a_j. On the other hand, setting the coefficients with even indexes to zero results in an antisymmetric polynomial. Following the same line of reasoning as the preceding subsection and employing the generalized binomial theorem that holds for noninteger powers as well, the coefficients c_j are identified to be the coefficients of the jth power of x in the power series expansion of $(1-x^2)^{N/2}$. This holds, regardless of the parity of N, as long as $d = 0$. Hence, we have

$$c_j|_{d=0} = 0, \qquad j \text{ odd}. \tag{17}$$

This shows that the resulting maximally flat filter of odd order satisfies the linear-phase condition (16).

C. Generalized Halfband Filters

The important class of halfband maximally flat filters are completely covered by Baher's filters. This includes the well-known linear-phase case [4] that is a special case of the recently studied nonsymmetric case [11]. Our simplified formula is identical to the transfer function of generalized (possibly nonsymmetric) halfband filters derived in [11] for $H_{N,(N/2)+1,d}(z)$, where N is an even integer, and d is a positive or negative integer that should be chosen so that its magnitude is less than $N/2$ and the parity of $(N/2)+d$ is odd. More precisely, we have [11]

$$\begin{aligned} &H_{N,(N/2)+1,d}(z) - H_{N,(N/2)+1,d}(-z) \\ &\quad = z^{-((N/2)+d)}, \quad \frac{N}{2}+d \text{ odd}, \quad N \text{ even}. \end{aligned}$$

D. Lagrange Interpolators

The versatility of Baher's filters may be further appreciated by considering the case $K = 0$. Absence of a zero at $z = -1$ implies that the resulting system might not be a frequency-selective filter. Here, we show that the system is in fact a Lagrange interpolator in this case.

As is pointed out in [10], the transfer function of a Lagrange interpolator $L(z)$ of order N is the solution to

$$L(z) = z^{-D}, \quad D = 0, 1, \cdots, N. \tag{18}$$

In other words, the interpolator should reduce to the trivial integer delay for integer values of delay parameter D. We now show that $H_{N,0,d}(z)$ satisfies the above set of equations if we set $D = (N/2)+d$. Specifically, we show the following.

Theorem 2: $H_{N,0,d}(z) = z^{-((N/2)+d)}$ for $(N/2)+d$ integer.

Proof: For integer values of $(N/2)+d$, the value of $(N/2)-d$ is also an integer, and hence, we have

$$\begin{aligned} H_{N,0,d}(z) &= \sum_{j=0}^{N}\sum_{i=0}^{j} (-1)^{j-i} \binom{\frac{N}{2}-d}{i} \binom{\frac{N}{2}+d}{j-i} \\ &\quad \cdot \left(\frac{1-z^{-1}}{2}\right)^j \left(\frac{1+z^{-1}}{2}\right)^{N-j} \\ &= \left(-\frac{1-z^{-1}}{2} + \frac{1+z^{-1}}{2}\right)^{(N/2)+d} \\ &\quad \cdot \left(\frac{1-z^{-1}}{2} + \frac{1+z^{-1}}{2}\right)^{(N/2)-d} \\ &= z^{-((N/2)+d)} \end{aligned}$$

■

$$H_{N,K,0}(z) = \begin{cases} \left(\frac{1+z^{-1}}{2}\right)^K \displaystyle\sum_{j=0}^{N-K/2} \binom{\frac{N}{2}}{j} \left(-\left(\frac{1-z^{-1}}{2}\right)^2\right)^j \\ \quad \cdot \left(\left(\frac{1+z^{-1}}{2}\right)^2\right)^{((N-K)/2)-j}, & K \text{ even} \\ \left(\frac{1+z^{-1}}{2}\right)^{K+1} \displaystyle\sum_{j=0}^{(N-K-1)/2} \binom{\frac{N}{2}}{j} \left(-\left(\frac{1-z^{-1}}{2}\right)^2\right)^j \\ \quad \cdot \left(\left(\frac{1+z^{-1}}{2}\right)^2\right)^{((N-K-1)/2)-j}, & K \text{ odd}. \end{cases} \tag{14}$$

Thus, we have proved that $H_{N,0,d}(z)$ is the Lagrange interpolator for all real values of d. It is noteworthy that representation of Lagrange interpolators in the Bernstein form using $H_{N,0,d}(z)$ is a novel expression for these fractional delay systems.

E. Impulse Response Coefficients

In this subsection, we give an explicit formula for the impulse response coefficients $h_k, k = 0, 1, \cdots, N$ of the universal maximally flat lowpass filter $H_{N,K,d}(z) = \sum_{k=0}^{N} h_k z^{-k}$. Expanding (10), and after some routine algebraic manipulations, we get (19), shown at the bottom of the page. The formula gives numerical values of h_k using the three parameters N, K, and d. It can be used to design all types of maximally flat systems listed in Table I.

IV. Generating Function and Multiplierless Arrays

The object of this part of the paper is to derive a generating function for the family of filters $\{H_{N,K,d}(z)\}$ and then develop multiplierless cellular structures based on it. We will see that such structures are possible only for special values of d for a given N. The general method for generation of cellular structures from a generating system is discussed elsewhere [17]. However, we provide a step-by-step exposition to the subject for the sake of self-containment.

A. Generating Function

Consider the family $\{H_{N,N-i,d}(z), i = 0, 1, \cdots\}$ of Baher's filters. The family includes all possible maximally flat FIR filters of order N with fixed group delay parameter d. The simplified formula presented in Section II can be directly used to compute the transfer functions for $0 \leq i \leq N$. Note that for values of i outside the interval $[0, N]$, i.e., for $K < 0$, usage of (10) generally results in a rational transfer function that is not an FIR system. Now, we can define a generating function of the form

$$G_{N,d}(x,z) = \sum_{i=0}^{\infty} H_{N,N-i,d}(z)x^i \tag{20}$$

for the family of filters. We wish to derive an explicit expression for $G_{N,d}(x,z)$. A common trick for derivation of closed forms for generating functions is evaluation of expressions of the form $(1-x)G_{N,d}(x,z)$. Thus, we continue as

$$\begin{aligned}(1-x)&G_{N,d}(x,z)\\ &= \sum_{i=0}^{\infty} H_{N,N-i,d}(z)x^i - \sum_{i=1}^{\infty} H_{N,N-i+1,d}(z)x^i\\ &= H_{N,N,d}(z) + \sum_{i=1}^{\infty}\\ &\quad \cdot (H_{N,N-i,d}(z) - H_{N,N-i+1,d}(z)), x^i.\end{aligned} \tag{21}$$

Using (10) and the definition of c_j, the right-hand side can be written as

$$\begin{aligned}(1-x)&G_{N,d}(x,z)\\ &= c_0\left(\frac{1+z^{-1}}{2}\right)^N + \sum_{i=1}^{\infty} c_i\left(\frac{1-z^{-1}}{2}\right)^i\\ &\quad \cdot \left(\frac{1+z^{-1}}{2}\right)^{N-i} \boldsymbol{x}_i\\ &= \sum_{i=0}^{\infty} c_i\left(\frac{1-z^{-1}}{2}\right)^i \left(\frac{1+z^{-1}}{2}\right)^{N-i} \boldsymbol{x}_i.\end{aligned} \tag{22}$$

From the binomial expansion theorem, it can be verified that the right-hand side of the above equation is the power series expansion of

$$\left(\frac{1+z^{-1}}{2} + x\,\frac{1-z^{-1}}{2}\right)^{(N/2)-d} \left(\frac{1+z^{-1}}{2} - x\,\frac{1-z^{-1}}{2}\right)^{(N/2)+d}.$$

Consequently, we obtain (23), shown at the bottom of the page, which is the closed-form expression for $G_{N,d}(x,z)$. The generating function is rational if and only if $(N/2)+d$ is an integer. In that case, the members of the family with $i > N$, i.e., those with negative values of K, become trivial systems of the form $z^{-((N/2)+d)}$. For noninteger values of $(N/2)+d$, however, the members with a negative K become IIR systems. In any case, the focus of this paper is the members whose K parameters are within the interval $[0, N]$.

$$h_k = \frac{\displaystyle\sum_{j=0}^{N-K}\sum_{p=0}^{k}\sum_{i=0}^{j}(-1)^{j-i+p}\binom{\frac{N}{2}-d}{i}\binom{\frac{N}{2}+d}{j-i}\binom{j}{p}\binom{N-j}{k-p}}{2^N}, \qquad k = 0, \cdots, N. \tag{19}$$

$$G_{N,d}(x,z) = \frac{\left(\frac{1+z^{-1}}{2} + x\,\frac{1-z^{-1}}{2}\right)^{(N/2)-d}\left(\frac{1+z^{-1}}{2} - x\,\frac{1-z^{-1}}{2}\right)^{(N/2)+d}}{1-x} \tag{23}$$

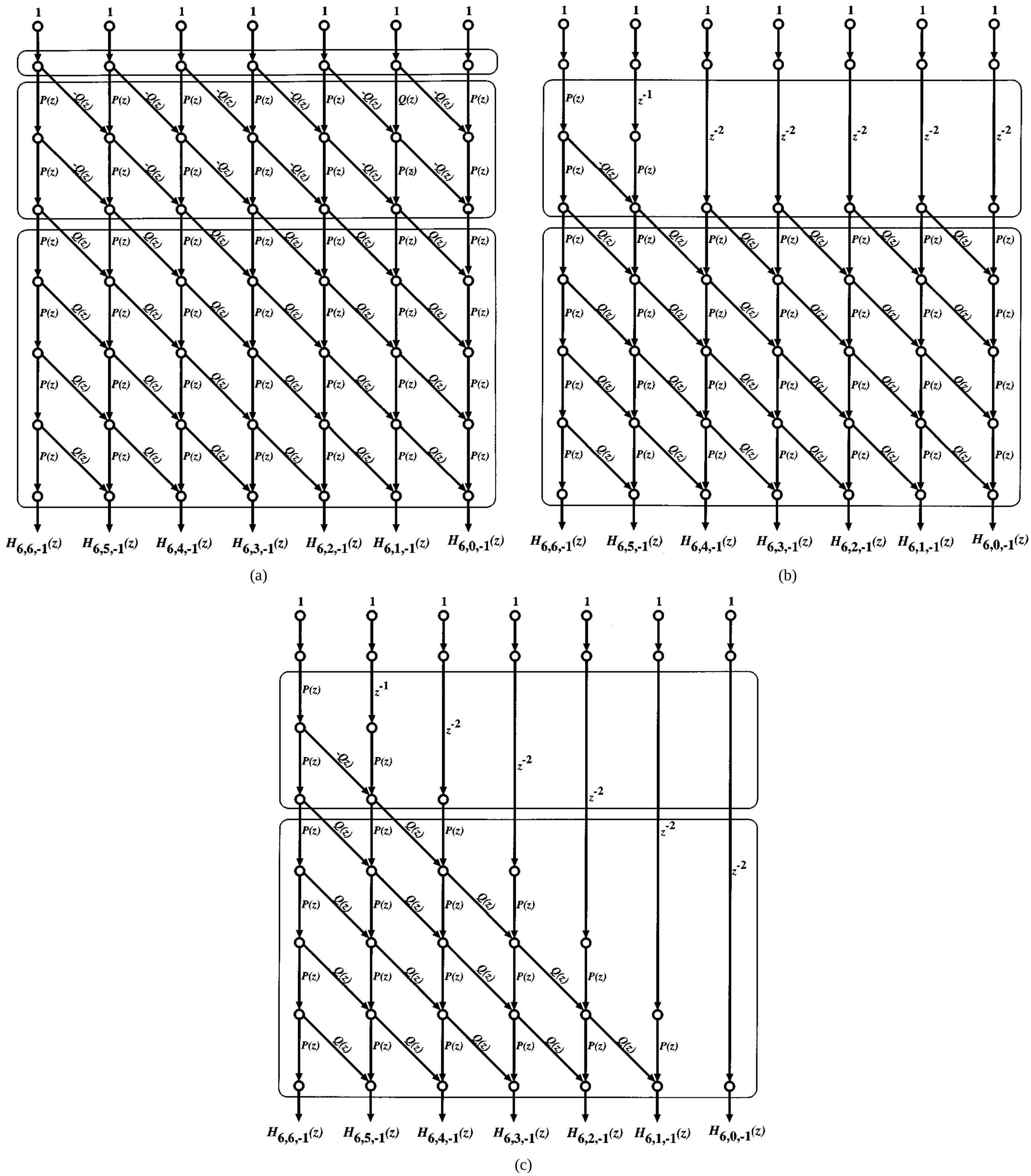


Fig. 2. Signal flowgraph for the family $\{H_{6,6-i,-1}(z),\, i = 0, \cdots, 6\}$.

B. Multiplierless Arrays

In the case of rational generating functions, a signal flowgraph for the family of filters can be obtained by writing down the difference equations obtained by plugging the spatial unit impulse signal into the discrete-time system $G_{N,\,d}(x, z)$. The indeterminate x should be viewed as the spatial delay operator. Since $G_{N,\,d}(x, z)$ is a cascade of simpler first-order subsystems, one can construct the signal flowgraph for each subsystem and then realize the overall system as a cascade. The response of subsystem $1/(1-x)$ to the unit impulse signal is the sequence $\{1, 1, 1, \cdots\}$. Thus ,we should cascade the signal

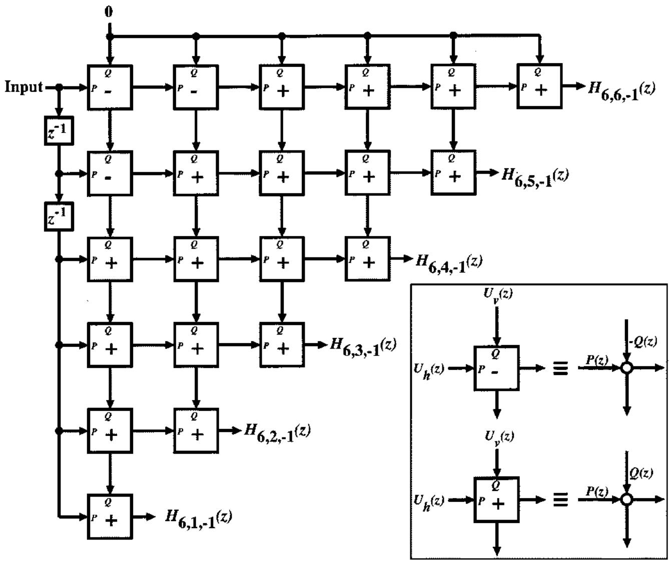


Fig. 3. Cellular multiplierless structure for the family $\{H_{6,6-i,-1}(z),\, i = 0, \cdots, 6\}$.

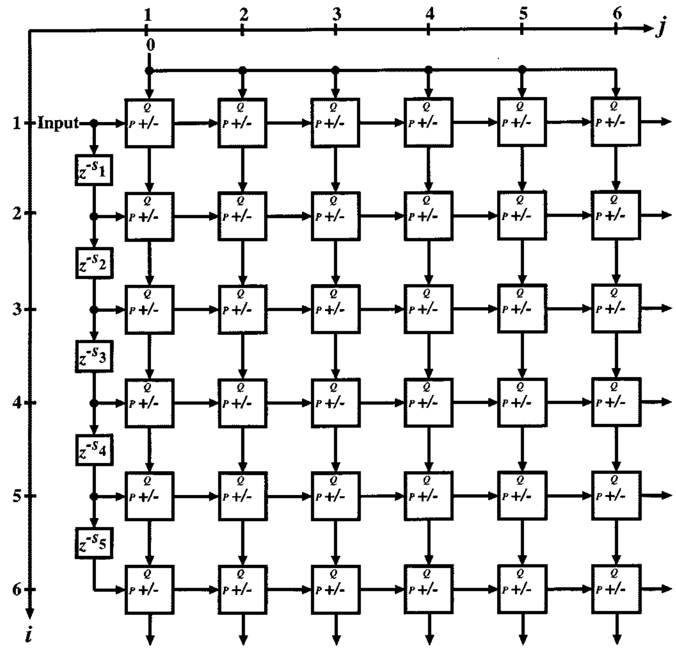


Fig. 4. General structure with configurable cells and boundary coefficients for realization of filters belonging to $\{H_{N,K,d}(z), (N/2) + d \text{ integer}\}$.

flowgraphs of the remaining $((N/2) - d) + ((N/2) + d) = N$ subsystems and feed them with the sequence $\{1, 1, 1, \cdots\}$. As a concrete example, let us derive the signal flowgraph for the case $N = 6$, $d = -1$. We need two subsystems of the form $(((1 + z^{-1})/2) - x((1 - z^{-1})/2))$ and four subsystems of the form $(((1 + z^{-1})/2) + x((1 - z^{-1})/2))$, which is a total of six cascaded subsystems. Fig. 2(a) presents the signal flowgraph of the family generated by $G_{6,-1}(x, z)$, where $P(z) = ((1+z^{-1})/2)$, and $Q(z) = P(z) - z^{-1} = ((1 - z^{-1})/2$. It can be seen that there exists a large number of redundant nodes in Fig. 2(a) that may be removed or simplified if the linear relation between polynomials $P(z)$ and $Q(z)$ is utilized. Fig. 2(b) shows the first stage of simplification in which the redundant nodes producing simple integer delays of the forms z^{-1} and z^{-2} are removed from the first group of two layers, and the branch transmittances are modified accordingly. The fully simplified signal flowgraph, which is obtained by economization of the nodes of the second group of four layers, is shown in Fig. 2(c).

The above development gives rise to a regular and multiplierless array structure for realization of the seven members belonging to $\{H_{6,6-i,-1}(z),\, i = 0, \cdots, 6\}$. An array structure consisting of two types of cells is illustrated in Fig. 3. The structure is obtained by integration of each node of the signal flowgraph of Fig. 2(c) together with its two incident branches into a single cell. The nodes at the boundaries of the signal flowgraph that possess only one incident branch are compensated by an auxiliary branch that receives zero input values. Adoption of this convention enhances the regularity of the resulting array structure. There are two types of cells. The cells labeled with a "$-$" execute the transfer function

$$Y(z) = U_h(z)\,P(z) - U_v(z)\,Q(z)$$

whereas the cells with a "$+$" label operate according to

$$Y(z) = U_h(z)\,P(z) + U_v(z)\,Q(z)$$

where $U_h(z)$ and $U_v(z)$ denote the z transforms of the horizontal and vertical inputs to the cells, respectively, and $Y(z)$ denotes the z transform of the cell output signal. Interestingly, the structure of Fig. 3 is scalable to realize filters of higher orders by adding extra cells in accordance with the generating function (23).

Provided that a configurable mesh array is available, one can obtain any member of the family $\{H_{N,\,K,\,d}(z), N/2+d \text{ integer}\}$ by applying appropriate boundary signals to a properly configured array structure. Fig. 4 shows a versatile configurable array for this purpose. The type of each cell ($+$ or $-$) is determined by the actual values of d and N. A procedure for determining the boundary signals and the type of cells is given below.

definitions
  J Number of columns
  I Number of rows
  s_i Value of s_i in z^{-s_i} in the delay chain
  $c_{i,\,j}$ Cell type
procedure *ConfigureArray*(N, K, d)
  $J \leftarrow K$
  $I \leftarrow N - K + 1$
  for $i = 1$ **to** $I - 1$ **do**
    if $i \leq \dfrac{N}{2} + d$ **then** $s_i \leftarrow 1$
    else $s_i \leftarrow 0$
  endfor
  for $i = 1$ **to** I **do**
    for $j = 1$ **to** J **do**
      if $i + j \leq \dfrac{N}{2} + d + 1$ **then** $c_{i,\,j} \leftarrow -$
      else $c_{i,\,j} \leftarrow +$
    endfor
  endfor
endprocedure

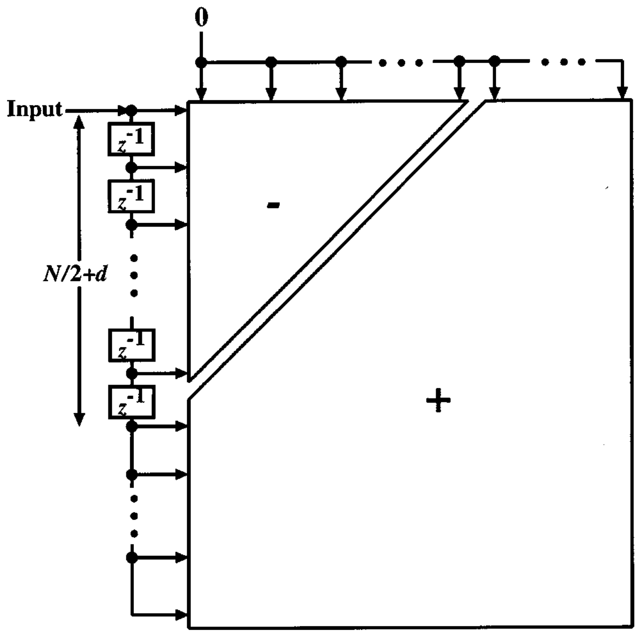


Fig. 5. Schematic arrangement of cells.

For the definition of the cell coordinates (i, j) see Fig. 4. In general, the cells get configured according to the schematic given in Fig. 5.

V. CONCLUSION

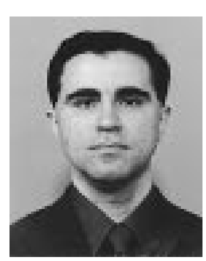
Halfband maximally flat FIR filters have found important applications in the areas of multirate systems and wavelet design in recent years. Nevertheless, there has been little work on the interrelation between different families of maximally flat filters, especially those with nonsymmetric impulse response coefficients. In this paper, we have shown that the family of maximally flat filters proposed by Baher is a universal family of maximally flat FIR filters. Specifically, we proved that linear-phase filters of even and odd orders, generalized halfband filters, and fractional delay systems known as Lagrange interpolators all belong to the family of Baher's filters and are obtained by particular choices of three parameters. Besides containing the aforementioned classes of special filters, Baher's filters are particularly useful because they yield tradeoff between the linearity of phase response and the width of transition band for frequency-selective systems.

A simplified formula has been presented for the transfer function of the filters and its equivalence with Baher's formula has been established. We proved the identity using a modern automatic technique for proving identities that involve binomial coefficients. A computer algebra system was used for automatic generation of the proof. A byproduct of the computer-generated proof is a three-term recurrence relation for the coefficients of the transfer function. The recurrence may find application in variable delay or variable order implementations where rapid update of the coefficients for new values of delay and/or order parameters is a highly desired feature.

We have also shown that the filters possess an explicit and simple generating function. Generating functions are useful tools that may lead to cellular systolic array structures for digital filters. For a special subclass of Baher's filters, a multiplierless array realization consisting of simple double-input, single-output cells is possible. A procedure for designing such array structures has been presented.

REFERENCES

[1] O. Herrmann, "On the approximation problem in nonrecursive digital filter design," *IEEE Trans. Circuit Theory*, vol. CT-18, pp. 411–413, 1971.
[2] J. A. Miller, "Maximally flat nonrecursive digital filters," *Electron. Lett.*, vol. 8, pp. 157–158, 1972.
[3] M. F. Fahmy, "Maximally flat nonrecursive digital filters," *Int. J. Circuit Theory Appl.*, vol. 4, pp. 311–313, 1976.
[4] C. Gumacos, "Weighting coefficients for certain maximally flat nonrecursive digital filters," *IEEE Trans. Circuits Syst.*, vol. CT-25, pp. 234–235, 1978.
[5] H. Baber, "FIR digital filters with simultaneous conditions on amplitude and delay," *Electron. Lett.*, vol. 18, pp. 296–297, 1982.
[6] L. R. Rajagopal and S. C. Dutta Roy, "Design of maximally-flat FIR filters using the Bernstein polynomial," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 1587–1590, Dec. 1987.
[7] I. W. Selesnick and S. Burrus, "Maximally flat low-pass FIR filters with reduced delay," *IEEE Trans. Circuits Syst. II*, vol. 45, pp. 53–68, Jan. 1998.
[8] N. Aikawa, N. Yabiku, and M. Sato, "Design method of odd order maximally flat FIR lowpass filter" (in Japanese), *Trans. Inst. Electron., Inform., Commun. Eng. A*, vol. J7-A, no. 4, pp. 762–768, 1992.
[9] ——, "Maximally flat FIR bandpass digital filter" (in Japanese), *Trans. Inst. Electron., Inform., Commun. Eng. A*, vol. J76-A, no. 10, pp. 1423–1430, 1993.
[10] T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay," *IEEE Signal Processing Mag.*, vol. 13, pp. 30–60, 1996.
[11] S. Samadi, A. Nishihara, and H. Iwakura, "Generalized half-band maximally flat FIR filters," in *Proc. ISCAS*, 1999.
[12] G. T. Cargo and O. Shisha, "The Bernstein form of a polynomial," *J. Res. Nat. Bur. Std. B*, vol. 70B, no. 1, pp. 79–81, 1966.
[13] R. T. Farouki and V. T. Rajan, "Algorithms for polynomials in Bernstein from," *Comput.-Aided Geo. Design*, vol. 5, pp. 1–26, 1988.
[14] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*. Reading, MA: Addison-Wesley, 1989.
[15] M. Petkovsek, H. S. Wilf, and D. Zeilberger, $A = B$. Wellesley, MA: A. K. Peters, 1996.
[16] K. Wegschaider, "Computer generated proofs of binomial multi-sum identities," Diploma thesis, RISC, J. Kepler Univ., Linz, Austria, May 1997.
[17] S. Samadi, A. Nishihara, and H. Iwakura, "Filter generating systems," *IEEE Trans. Circuits Syst. II*, vol. 47, pp. 214–221, Mar. 2000.



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